5.5 Substitution technique:

$$\int f(\underbrace{g(x)}) \underbrace{g'(x)dx} = \int f(u) \cdot du$$

Chain rule in "reverse"

$$u = g(x) \longrightarrow \frac{du}{dx} = g'(x) \longrightarrow du = g'(x)dx$$

Eg: Evaluate $\int \underline{3x^2} \cos(x^3) \underline{dx}$

Let $u = x^3 \longrightarrow du = \underline{3x^2 dx}$

New: $\int \cos(u)\, du$ where $u = x^3$, $du = 3x^2 dx$

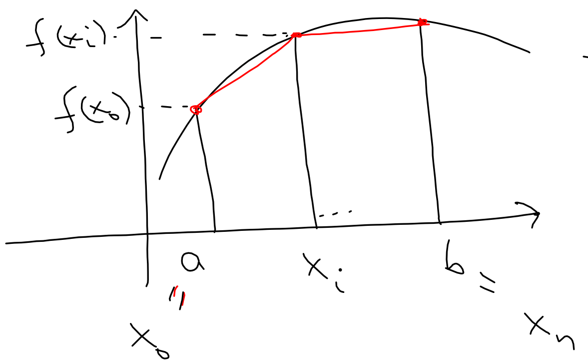
Hence $\int \cos u \, du = \sin(u) + C = \sin x^3 + C$

Verify: $\frac{d}{dx}\left(\sin x^3\right) = 3x^2 \cos(x^3)$ ✓

5.6 Numerical techniques of integration

These techniques are used to approximate the area under the curve of a continuous function over a well-defined domain

1 Trapezoidal rule



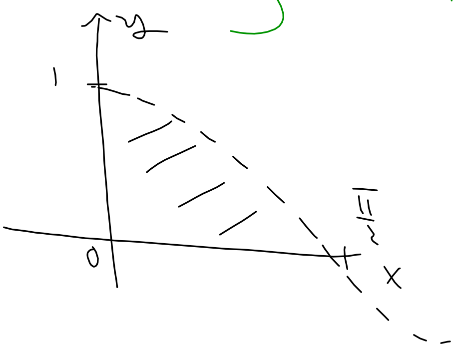
$$\lim_{\|\Delta x\| \to 0} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx$$

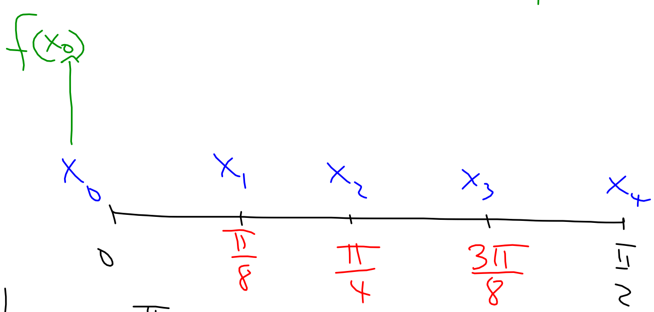
$$\frac{b-a}{n} = \Delta x$$

$$\boxed{\text{Area} \approx \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]}$$

Eg: Find the area of the region above the x-axis that is bounded by $x = 0$, $x = \frac{\pi}{2}$ and the curve $y = \cos x$

* using the trapezoidal rule: with $n = 4$





base: $\Delta x = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{4} = \frac{\pi}{8}$

Endpoints: $0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}$

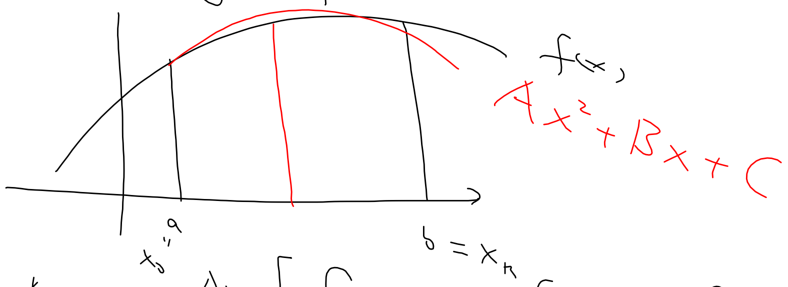
heights: $\cos(0)$; $\cos\left(\frac{\pi}{8}\right)$; $\cos\left(\frac{\pi}{4}\right)$; $\cos\left(\frac{3\pi}{8}\right)$; $\cos\left(\frac{\pi}{2}\right)$

$$\text{Area} = \frac{\frac{\pi}{8}}{2}\left[1 + 2\cos\left(\frac{\pi}{8}\right) + 2\cos\left(\frac{\pi}{4}\right) + 2\cos\left(\frac{3\pi}{8}\right) + 0\right]$$

$$\boxed{\simeq 0.9871 \text{ unit}^2}$$

② Simpson's rule

Here we use instead of linear functions, quadratic functions to approximate the the height of the blocks.



n = even!

$$\text{Area} \simeq \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)\Big]$$

Example: apply the simpson's rule to the previous region

We have:

$$\text{Area} \simeq \frac{\frac{\pi}{8}}{3}\left[1 + 4\left(\cos\frac{\pi}{8}\right) + 2\left(\cos\left(\frac{\pi}{4}\right)\right) + 4\left(\cos\left(\frac{3\pi}{8}\right)\right) + 0\right]$$

$$\boxed{\simeq 1.0001 \text{ unit}^2}$$

Remark:

The exact value of $\int_0^{\frac{\pi}{2}} \cos x \, dx$

$$A = \Big[\sin x\Big]_0^{\frac{\pi}{2}} = \sin\left(\tfrac{\pi}{2}\right) - \sin(0) = \boxed{1 \text{ unit}^2}$$

Example

Approximate using $n = 4$ the following $\int_0^1 x^2\,dx$

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$$

Endpoints: $0;\ \frac{1}{4};\ \frac{1}{2};\ \frac{3}{4};\ 1$

heights: $0^2;\ \left(\frac{1}{4}\right)^2;\ \left(\frac{1}{2}\right)^2;\ \left(\frac{3}{4}\right)^2;\ (1)^2$

ⓐ Trapezoidal:

$$\text{Area} \simeq \frac{1}{8}\left[0^2 + 2\left(\frac{1}{4}\right)^2 + 2\left(\frac{1}{2}\right)^2 + 2\left(\frac{3}{4}\right)^2 + (1)^2\right]$$

$$\simeq .3438$$

ⓑ Simpson's rule:

$$\text{Area} \simeq \frac{1}{12}\left[0^2 + 4\left(\frac{1}{4}\right)^2 + 2\left(\frac{1}{2}\right)^2 + 4\left(\frac{3}{4}\right)^2 + (1)^2\right]$$

$$\simeq .33\overline{33}$$

ⓒ Exact value:

$$A = \int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1 = \boxed{\frac{1}{3}\ \text{unit}^2}$$

Evaluate $\int \frac{x}{(\underbrace{2-2x^2}_{})^3}\,dx$

$u = 2-2x^2 \longrightarrow du = -4x\,dx$

$\int \frac{x\,dx}{u^3}$ $\longrightarrow -\frac{1}{4}du = x\,dx$

!

incomplete substitution!

$\longrightarrow \int \frac{-\frac{1}{4}du}{u^3} = -\frac{1}{4}\int \frac{du}{u^3} = -\frac{1}{4}\int u^{-3}\,du$

$= -\frac{1}{4}\left(\frac{1}{-2}u^{-2}\right) + C$

$= \frac{u^{-2}}{8} + C = \frac{1}{8u^2} + C$

$= \frac{1}{8(2-2x^2)^2} + C$

Example: Evaluate $\int \frac{1}{\underbrace{7-7x}_{u}} dx$

$u = 7-7x \longrightarrow du = -7dx$

$\longrightarrow -\frac{1}{7} du = dx$

$$\int \frac{1}{u}\left(-\frac{1}{7}du\right) = -\frac{1}{7}\int \frac{du}{u} = -\frac{1}{7}\int u^{-1} du$$

$$= -\frac{1}{7}\ln|u| + C$$

$$= -\frac{1}{7}\ln|7-7x| + C$$

Evaluate $\int \frac{1}{x \ln x^6} dx = \int \frac{1}{x} \cdot \frac{1}{\ln x^6} dx$ $\qquad \frac{dx}{x}$

$u = \ln x^6 \quad du = \frac{6}{x} dx \qquad (\ln u)' = \frac{u'}{u}$

$\frac{1}{6} du = \frac{dx}{x}$

$\int \frac{1}{u}\left(\frac{1}{6} du\right) = \frac{1}{6} \int \frac{1}{u} du$

$= \frac{1}{6} \ln|u| = \boxed{\frac{1}{6} \ln \left| \ln x^6 \right| + C}$